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A note on a Carlson-type inequality for the Sugeno integral

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ABSTRACT

In this short note, we present a general version of Carlson's inequality for the Sugeno integral, which generalizes some recent results obtained by others.

1. Introduction

Fuzzy measures and the fuzzy integral (also known as the Sugeno integral) can be used for modeling problems in non-deterministic environments. Since Sugeno [1] initiated research on the fuzzy measure and fuzzy integral, this area has been widely developed, and a wide variety of topics have been investigated (see, e.g., [2,3] and references therein).

Recently, the study of fuzzy integral inequalities has become a popular topic in the fuzzy society [4–9]. For example, Flores-Franulič and Román-Flores [5] provided some Chebyshev-type inequalities for the Sugeno integral of continuous and strictly monotone real functions based on the Lebesgue measure. Later on, Ouyang et al. generalized the fuzzy Chebyshev-type inequalities to the case of the arbitrary fuzzy measure-based Sugeno integral [6,10] and then further generalized them to the so-called seminormed fuzzy integral [7].

Very recently, Caballero and Sadarangani [11] proved a Carlson inequality for the Sugeno integral. In fact, they proved the following result.

Theorem 1.1. *Let $f\colon [0,1] \to [0,\infty]$ be a non-decreasing function and μ be the Lebesgue measure on R. Then*

$$(S)\int_0^1 f d\mu \le \sqrt{2}\left((S)\int_0^1 x^2 f^2 d\mu\right)^{\frac{1}{4}} \cdot \left((S)\int_0^1 f^2 d\mu\right)^{\frac{1}{4}}. \tag{1.1}$$

Notice that this result only deals with a special case, i.e., the Lebesgue measure-based Sugeno integral. Hence there is a natural question: Does this inequality holds for an arbitrary fuzzy measure-based Sugeno integral? This is an interesting question, and we will give an affirmative answer for it in this paper. In the next section, we give some basic concepts and known results that will be used in this paper, and then we answer this question in Section 3.

2. Preliminaries

As usual, we denote by $\mathbf{R}$ the set of real numbers. Let X be a non-empty set, and let $\mathcal{F}$ be a σ-algebra of subsets of X. Let $\overline{\mathbf{R}}_+$ denote $[0,+\infty]$. Throughout this paper, all considered subsets are supposed to belong to $\mathcal{F}$.

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Definition 2.1 (*Ralescu and Adams [12]*). A set function $\mu : \mathcal{F} \to \overline{\mathbf{R}_+}$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$;
(FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$;
(FM3) $A_1 \subset A_2 \subset \cdots$ implies that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$; and
(FM4) $A_1 \supset A_2 \supset \cdots$, and $\mu(A_1) < +\infty$ imply that $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

When μ is a fuzzy measure, the triple $(X, \mathcal{F}, \mu)$ is called a fuzzy measure space.

Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. By $\mathcal{F}_+(X)$ we denote the set of all non-negative $\mathcal{F}$-measurable functions with respect to $\mathcal{F}$. In what follows, all considered functions belong to $\mathcal{F}_+^{\mu}(X)$. Let f be a non-negative real-valued function defined on X. We will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α for $\alpha \geq 0$. Clearly, F_α is non-increasing with respect to α, i.e., $\alpha \leq \beta$ implies that $F_\alpha \supseteq F_\beta$.

Definition 2.2 (*Pap [2], Wang and Klir [3]*). Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and let $A \in \mathcal{F}$. The Sugeno integral of f on A, with respect to the fuzzy measure μ, is defined by

$$(S)\int_A f d\mu = \bigvee_{\alpha\geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When $A = X$,

$$(S)\int_X f d\mu = (S)\int f d\mu = \bigvee_{\alpha\geq 0} (\alpha \wedge \mu(F_\alpha)).$$

Some basic properties of the Sugeno integral are summarized in [2,3]. We cite some of them as follows.

Theorem 2.1 (*Pap [2], Wang and Klir [3]*). *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. Then*

(i) $\mu(A \cap F_\alpha) \geq \alpha \Longrightarrow (S)\int_A f d\mu \geq \alpha$;
(ii) $\mu(A \cap F_\alpha) \leq \alpha \Longrightarrow (S)\int_A f d\mu \leq \alpha$;
(iii) $(S)\int_A f d\mu = \alpha \Longleftrightarrow \mu(A \cap F_\alpha) \geq \alpha \geq \mu(A \cap F_{\alpha+})$, *where* $\mu(A \cap F_{\alpha+}) = \lim_{\varepsilon\to 0} \mu(A \cap F_{\alpha+\varepsilon})$;
(iv) $(S)\int_A f d\mu < \alpha \Longleftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap F_\gamma) < \alpha$;
(v) $(S)\int_A f d\mu > \alpha \Longleftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap F_\gamma) > \alpha$; *and*
(vi) *if* $f \leq g$, *then* $(S)\int f d\mu \leq (S)\int g d\mu$.

Recall that two functions $f, g: X \to \mathbf{R}$ are said to be comonotone if, for all $(x, y) \in X^2$, $(f(x) - f(y))(g(x) - g(y)) \geq 0$. Clearly, if f and g are comonotone, then, for any real numbers p, q, either $F_p \subset G_q$ or $G_q \subset F_p$. It is well known that the Sugeno integral is comonotone maxitive (see [13]), i.e., if f and g are comonotone, then

$$(S)\int f \vee g d\mu = \left((S)\int f d\mu\right) \bigvee \left((S)\int g d\mu\right). \tag{2.1}$$

Notice that (2.1) can be regarded as the comonotone commuting property of the Sugeno integral with respect to the maximum. For a complete investigation of comonotone commuting property of the Sugeno integral, we recommend a recent paper of Ouyang and Mesiar [14]. They proved that there are only 18 classes of operator $\star$: $[0, \infty]^2 \to [0, \infty]$ such that

$$(S)\int f \star g d\mu = \left((S)\int f d\mu\right) \star \left((S)\int g d\mu\right). \tag{2.2}$$

3. Main results

Before presenting our main result, we give two lemmas.

Lemma 3.1. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, let $A \in \mathcal{F}$, and let $f: X \to R$ be a measurable function such that $(S)\int_A f d\mu \leq 1$. Then, for any $s \geq 1$, we have*

$$(S)\int_A f^s d\mu \geq \left((S)\int_A f d\mu\right)^s. \tag{3.1}$$

Proof. Let $(S)\int_A f d\mu = r \leq 1$. Then, by (iii) of Theorem 2.1, we have $\mu(A \cap F_r) \geq r$. Hence, for $s \geq 1$,

$$\mu(A \cap \{x | f^s(x) \geq r^s\}) = \mu(A \cap F_r) \geq r \geq r^s. \tag{3.2}$$

Now, (i) of Theorem 2.1 implies that $(S)\int_A f^s d\mu \geq r^s$. □

Lemma 3.2 ([6]). *Let $f, g \in \mathcal{F}^+(X)$, and let μ be an arbitrary fuzzy measure such that both $(S)\int_A f d\mu$ and $(S)\int_A g d\mu$ are finite. And let $\star: [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$(S)\int_A f \star g d\mu \geq \left((S)\int_A f d\mu\right) \star \left((S)\int_A g d\mu\right) \tag{3.3}$$

holds.

Note 3.1. Note that, if both $(S)\int_A f d\mu$ and $(S)\int_A g d\mu$ are bounded by a constant c, then the requirement $\star|_{[0,c]^2} \leq \min$ is enough to ensure the validity of Lemma 3.2.

Theorem 3.1. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, let $A \in \mathcal{F}$, and let $f_i: X \to R$, $i = 1, 2, 3$ be measurable functions such that $(S)\int_A f_i d\mu \leq 1$. If any two functions of f_i, $i = 1, 2, 3$ are comonotone, then, for any $p, q \geq 1$, we have*

$$(S)\int_A f_1 d\mu \leq \frac{1}{\sqrt{C}} \left((S)\int_A f_1^p f_2^p d\mu\right)^{\frac{1}{2p}} \left((S)\int_A f_1^q f_3^q d\mu\right)^{\frac{1}{2q}}, \tag{3.4}$$

where $C = \left((S)\int_A f_2 d\mu\right)\left((S)\int_A f_3 d\mu\right)$.

Proof. First, we show that $(S)\int_A f_1 d\mu \leq 1$ and $(S)\int_A f_2 d\mu \leq 1$ imply that $(S)\int_A f_1 f_2 d\mu \leq 1$. In fact, by (iii) of Theorem 2.1, we have $\mu(A \cap F_{1+}^{(1)}) \leq 1$ and $\mu(A \cap F_{1+}^{(2)}) \leq 1$, where $\mu(A \cap F_{1+}^{(i)}) = \lim_{\varepsilon \to 0} \mu(A \cap \{x | f_i(x) \geq 1 + \varepsilon\})$. Thus the comonotonicity of f_1 and f_2 implies that

$$\mu\left(A \bigcap (F_{1+}^{(1)} \cup F_{1+}^{(2)})\right) \leq 1.$$

Noting the fact that $\{x | f_1(x) f_2(x) > 1\} \subset \{x | f_1(x) > 1\} \cup \{x | f_2(x) > 1\}$, we have that

$$\mu\left(A \cap \{x | f_1(x) f_2(x) \geq 1 + \varepsilon\}\right) \leq 1,$$

for any $\varepsilon > 0$. Again, by (iii) of Theorem 2.1, we conclude that $(S)\int_A f_1 f_2 d\mu \leq 1$.

Now, by Lemma 3.1, for $p \geq 1$, we have

$$\left((S)\int_A f_1 f_2 d\mu\right)^p \leq (S)\int_A f_1^p f_2^p d\mu. \tag{3.5}$$

Similarly, for $q \geq 1$, it holds that

$$\left((S)\int_A f_1 f_3 d\mu\right)^q \leq (S)\int_A f_1^q f_2^q d\mu. \tag{3.6}$$

Thus

$$\left((S)\int_A f_1 f_2 d\mu\right)\left((S)\int_A f_1 f_3 d\mu\right) \leq \left((S)\int_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \left((S)\int_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}. \tag{3.7}$$

Since the usual product satisfies $\cdot|_{[0,1]^2} \leq \min$, by Lemma 3.2,

$$(S)\int_A f_1 f_2 d\mu \geq \left((S)\int_A f_1 d\mu\right)\left((S)\int_A f_2 d\mu\right) \tag{3.8}$$

and

$$(S)\int_A f_1 f_3 d\mu \geq \left((S)\int_A f_1 d\mu\right)\left((S)\int_A f_3 d\mu\right). \tag{3.9}$$

Thus

$$\left((S)\int_A f_1 d\mu\right)^2 \leq \frac{1}{\left((S)\int_A f_2 d\mu\right)\left((S)\int_A f_3 d\mu\right)} \left((S)\int_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \left((S)\int_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}. \tag{3.10}$$

If we denote $C = \left((S)\int_A f_2 d\mu\right)\left((S)\int_A f_3 d\mu\right)$, then we get the desired result:

$$(S)\int_A f_1 d\mu \leq \frac{1}{\sqrt{C}} \left((S)\int_A f_1^p f_2^p d\mu\right)^{\frac{1}{2p}} \left((S)\int_A f_1^q f_3^q d\mu\right)^{\frac{1}{2q}}. \quad \square$$

Example 3.1. Let $X=[0,2]$ and $\mu=\lambda^2$, where λ is the Lebesgue measure. Let

$$f_i(x)=\begin{cases} x & x\in\left[0,\frac{3}{2}\right], \\ 2-x & x\in\left(\frac{3}{2},2\right]. \end{cases}$$

Then $(S)\int_X f_i(x)d\mu=\frac{4-\sqrt{7}}{2}$ and $(S)\int_X f_1(x)f_2(x)d\mu=(S)\int_X f_1(x)f_3(x)d\mu=\frac{9}{16}$. Thus, for $p=q=1$, it holds that

$$(S)\int_X f_1 d\mu\le\frac{1}{\sqrt{C}}\left((S)\int_X f_1f_2d\mu\right)^{\frac{1}{2}}\left((S)\int_X f_1f_3d\mu\right)^{\frac{1}{2}},$$

where $C=\left((S)\int_X f_2d\mu\right)\left((S)\int_X f_3d\mu\right)$.

Let $f_2(x)=x$ and $f_3(x)=1,\ \forall x\in X$, and let $p=q=2$. Then we have the following corollary, which is the main result of [11].

Corollary 3.1. *Let $f:[0,1]\to[0,\infty)$ be a non-decreasing function, and let μ be the Lebesgue measure on R. Then*

$$(S)\int_0^1 f(x)d\mu\le\sqrt{2}\left((S)\int_0^1 x^2f^2(x)d\mu\right)^{\frac{1}{4}}\left((S)\int_0^1 f^2(x)d\mu\right)^{\frac{1}{4}}. \tag{3.11}$$

If we let $f_2(x)=\frac{1}{x}$ and $f_3(x)=1,\ \forall x\in X$, and let $p=q=2$, then we can obtain the following interesting result.

Corollary 3.2. *Let $f:[0,1]\to[0,\infty)$ be a non-increasing function, and let μ be the Lebesgue measure on R. Then*

$$(S)\int_0^1 f(x)d\mu\le\left((S)\int_0^1 \frac{f^2(x)}{x^2}d\mu\right)^{\frac{1}{4}}\left((S)\int_0^1 f^2(x)d\mu\right)^{\frac{1}{4}}. \tag{3.12}$$

The following example shows that, if the comonotonicity of $f_i,\ i=1,2,3$ is absent, then the conclusion of Theorem 3.1 may not be true.

Example 3.2. Let $X=[0,2]$ and $f_1(x)=x$, $f_2(x)=f_3(x)=2-x$, and let $p=q=1$. If μ is the Lebesgue measure, then

$$(S)\int_0^2 f_i(x)d\mu=\bigvee_{\alpha\in[0,2]}(\alpha\wedge 2-\alpha)=1,\quad i=1,2,3$$

and

$$(S)\int_0^2 f_1(x)f_2(x)d\mu=(S)\int_0^2 f_1(x)f_3(x)d\mu\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge 2\sqrt{1-\alpha}\right)=2\sqrt{2}-1.$$

Thus

$$(S)\int_0^2 f_1(x)d\mu=1>2\sqrt{2}-1=\frac{1}{\sqrt{C}}\left((S)\int_0^2 f_1(x)f_2(x)d\mu\right)^{\frac{1}{2}}\left((S)\int_0^2 f_1(x)f_3(x)d\mu\right)^{\frac{1}{2}},$$

which violates (3.4).

The following example shows that, if we omit the condition $(S)\int_A f_i d\mu\le 1$, then Theorem 3.1 may not hold.

Example 3.3. Let $X=[0,10]$, $f_i(x)=x,\ i=1,2,3$, let μ be the Lebesgue measure, and let $p=q=1$. Then

$$(S)\int_X f_i(x)d\mu=\bigvee_{\alpha\in[0,10]}(\alpha\wedge 10-\alpha)=5,$$

and thus $C=25$. But

$$(S)\int_X f_1(x)f_2(x)d\mu=(S)\int_X f_1(x)f_3(x)d\mu=(S)\int_X x^2d\mu=\frac{21-\sqrt{41}}{2},$$

which implies that

$$(S)\int_X f_1(x)d\mu = 5 > \frac{21-\sqrt{41}}{10} = \frac{1}{\sqrt{C}}\left((S)\int_X f_1(x)f_2(x)d\mu\right)^{\frac{1}{2}}\left((S)\int_X f_1(x)f_3(x)d\mu\right)^{\frac{1}{2}},$$

which contradicts (3.4).

After submission of this paper, we noted that a similar result has been obtained by Wang and Bai [15]. In fact, under some similar conditions, Wang and Bai proved the following result:

$$(S)\int_A f d\mu \le \frac{1}{K}\left((S)\int_A f^p g^p d\mu\right)^{\frac{1}{p+q}}\left((S)\int_A f^q h^q d\mu\right)^{\frac{1}{p+q}},$$

where $K = \left((S)\int_A g d\mu\right)^{\frac{p}{p+q}}\left((S)\int_A h d\mu\right)^{\frac{q}{p+q}}$. One can see that the results of our paper and the above inequality are complementary to each other.

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